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MATHEMATICS
HIGHER LEVEL
PAPER 1

Candidate session number

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Monday 11 November 2013 (afternoon)

2 hours

Examination code

8	8	1	3	–	7	2	0	1

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

14 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 5]*

The cubic polynomial $3x^3 + px^2 + qx - 2$ has a factor $(x+2)$ and leaves a remainder 4 when divided by $(x+1)$. Find the value of p and the value of q.

2. *[Maximum mark: 6]*

The discrete random variable X has probability distribution:

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	a

(a) Find the value of a. *[1]*

(b) Find $\mathrm{E}(X)$. *[2]*

(c) Find $\mathrm{Var}(X)$. *[3]*

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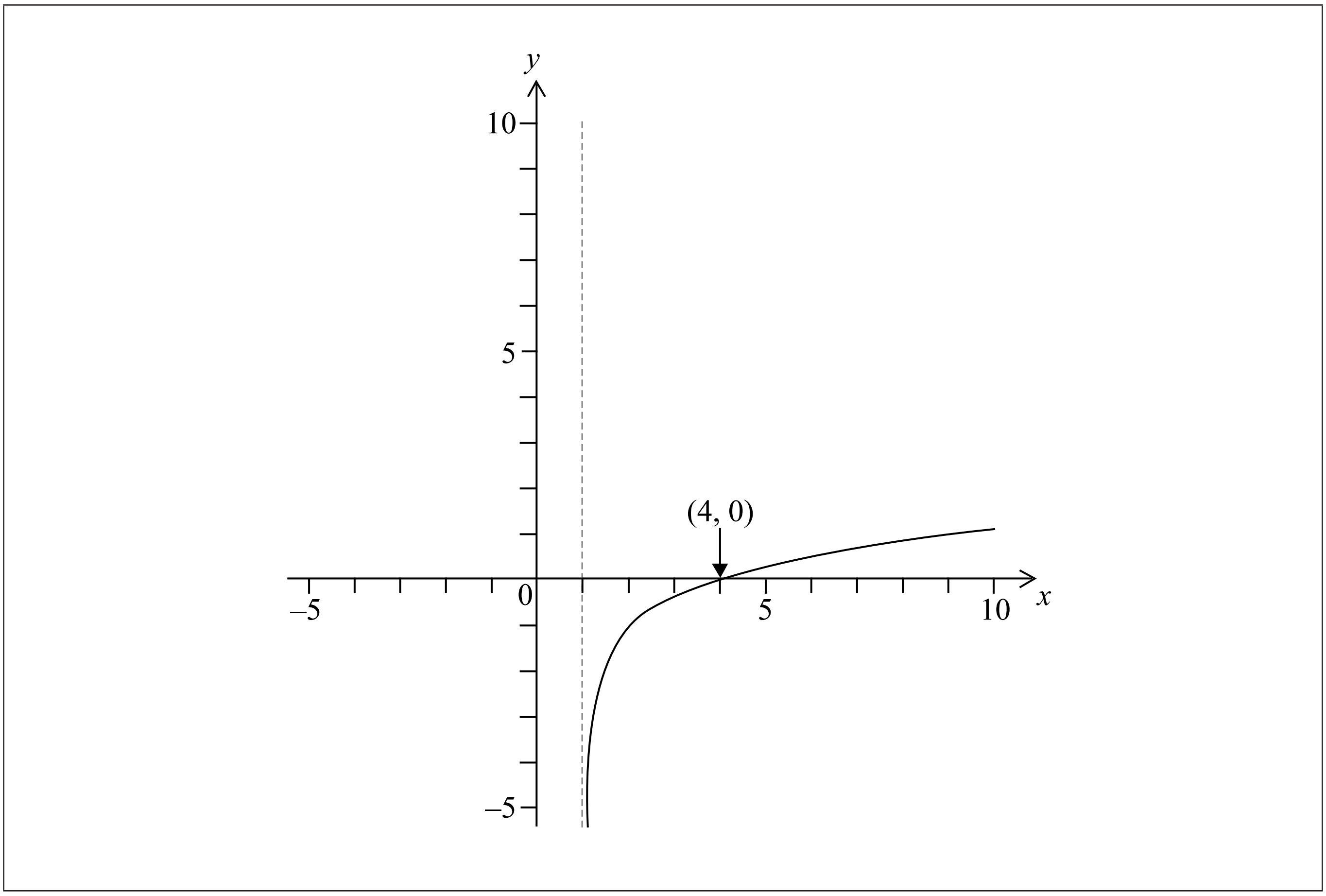
3. *[Maximum mark: 7]*

The diagram below shows a sketch of the graph of $y = f(x)$.



(a) Sketch the graph of $y = f^{-1}(x)$ on the same axes. *[2]*

(b) State the range of f^{-1}. *[1]*

(c) Given that $f(x) = \ln(ax + b)$, $x > 1$, find the value of a and the value of b. *[4]*

(This question continues on the following page)

(Question 3 continued)

4. *[Maximum mark: 5]*

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & a(a+1) \\ 1 & b(b+1) \end{pmatrix}$, $a \neq b$. Given that $\boldsymbol{A}$ is singular, find the value of $a+b$.

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5. *[Maximum mark: 7]*

A curve has equation $x^3y^2 + x^3 - y^3 + 9y = 0$. Find the coordinates of the three points on the curve where $\frac{dy}{dx} = 0$.

6. *[Maximum mark: 7]*

Prove by mathematical induction that $n^3 + 11n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

7. *[Maximum mark: 7]*

The sum of the first two terms of a geometric series is 10 and the sum of the first four terms is 30.

(a) Show that the common ratio r satisfies $r^2 = 2$. *[4]*

(b) Given $r = \sqrt{2}$

(i) find the first term;

(ii) find the sum of the first ten terms. *[3]*

8. *[Maximum mark: 8]*

(a) Prove the trigonometric identity $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$. *[4]*

(b) Given $f(x) = \sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)$, $x \in [0, \pi]$, find the range of f. *[2]*

(c) Given $g(x) = \csc\left(x+\frac{\pi}{6}\right)\csc\left(x-\frac{\pi}{6}\right)$, $x \in [0, \pi]$, $x \neq \frac{\pi}{6}$, $x \neq \frac{5\pi}{6}$, find the range of g. *[2]*

9. *[Maximum mark: 8]*

Solve the following equations:

(a) $\log_2(x-2) = \log_4\left(x^2 - 6x + 12\right)$; *[3]*

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$. *[5]*

*Do **NOT** write solutions on this page.*

SECTION B

*Answer **all** questions in the answer booklet provided. Please start each question on a new page.*

10. *[Maximum mark: 20]*

The function f is given by $f(x) = x\mathrm{e}^{-x}$ $(x \geq 0)$.

(a) (i) Find an expression for $f'(x)$.

(ii) Hence determine the coordinates of the point A, where $f'(x) = 0$. *[3]*

(b) Find an expression for $f''(x)$ and hence show the point A is a maximum. *[3]*

(c) Find the coordinates of B, the point of inflexion. *[2]*

(d) The graph of the function g is obtained from the graph of f by stretching it in the x-direction by a scale factor 2.

(i) Write down an expression for $g(x)$.

(ii) State the coordinates of the maximum C of g.

(iii) Determine the x-coordinates of D and E, the two points where $f(x) = g(x)$. *[5]*

(e) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, showing clearly the points A, B, C, D and E. *[4]*

(f) Find an exact value for the area of the region bounded by the curve $y = g(x)$, the x-axis and the line $x = 1$. *[3]*

*Do **NOT** write solutions on this page.*

11. *[Maximum mark: 20]*

Consider the points $A(1, 0, 0)$, $B(2, 2, 2)$ and $C(0, 2, 1)$.

(a) Find the vector $\vec{CA} \times \vec{CB}$. *[4]*

(b) Find an exact value for the area of the triangle ABC. *[3]*

(c) Show that the Cartesian equation of the plane Π_1, containing the triangle ABC, is $2x + 3y - 4z = 2$. *[3]*

A second plane Π_2 is defined by the Cartesian equation $\Pi_2: 4x - y - z = 4$. L_1 is the line of intersection of the planes Π_1 and Π_2.

(d) Find a vector equation for L_1. *[5]*

A third plane Π_3 is defined by the Cartesian equation $16x + \alpha y - 3z = \beta$.

(e) Find the value of α if all three planes contain L_1. *[3]*

(f) Find conditions on α and β if the plane Π_3 does **not** intersect with L_1. *[2]*

*Do **NOT** write solutions on this page.*

12. *[Maximum mark: 20]*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Use De Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}^+$. *[2]*

(b) Expand $\left(z + z^{-1}\right)^4$. *[1]*

(c) Hence show that $\cos^4\theta = p\cos 4\theta + q\cos 2\theta + r$, where p, q and r are constants to be determined. *[4]*

(d) Show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$. *[3]*

(e) Hence find the value of $\int_0^{\frac{\pi}{2}} \cos^6\theta \, \mathrm{d}\theta$. *[3]*

The region S is bounded by the curve $y = \sin x\cos^2 x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

(f) S is rotated through 2π radians about the x-axis. Find the value of the volume generated. *[4]*

(g) (i) Write down an expression for the constant term in the expansion of $\left(z + z^{-1}\right)^{2k}$, $k \in \mathbb{Z}^+$.

(ii) Hence determine an expression for $\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \, \mathrm{d}\theta$ in terms of k. *[3]*

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